

MATCHING REALIZATION OF $U_q(sl_{n+1})$ OF HIGHER RANK IN THE QUANTUM WEYL ALGEBRA $\mathcal{W}_q(2n)$

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ABSTRACT. In the paper, we further realize the higher rank quantized universal enveloping algebra $U_q(sl_{n+1})$ as certain quantum differential operators in $\mathcal{W}_q(2n)$ defined over the quantum divided power algebra $\mathcal{A}_q(n)$ of rank n . We give the quantum differential operators realization for both the simple root vectors and the non-simple root vectors of $U_q(sl_{n+1})$. The nice behavior of the quantum root vectors formulas under the action of the Lusztig symmetries once again indicates that our realization model is naturally matched.

1. INTRODUCTION AND PRELIMINARIES

1.1. It is well known that there exist quantum analogues $U_q(\mathfrak{g})$ for Lie algebras \mathfrak{g} with (generalized) Cartan matrices. But how to quantize the Lie algebras of Cartan type such that their 0-component parts coincide with the standard Drinfeld-Jimbo quantum groups is still an open problem. This is a hard question aimed by the first author when he was a Humboldt research fellow almost twenty years ago. On the other hand, fortunately, a series work on the modular quantizations of the modular simple restricted Lie algebras of Cartan type have been done in recent years via the Jordanian twists and modular reductions, and these provide us with some new non-pointed Hopf algebras of prime-power dimensions defined over a field of positive characteristic, see [6], [7], [14]. This will be significant if one considers the Kaplansky's 10 questions proposed in early 1975 which are related to classifying some finite-dimensional Hopf algebras in some sense, and compares with the seminal work on classifying finite-dimensional complex pointed Hopf algebras due to Andruskiewitsch-Schneider [1] in 2010. Here we would like to point out that these modular quantizations are non-pointed and do not belong to the standard ones in the sense of Drinfeld-Jimbo type.

As the Cartan type Lie algebras contain four series W, S, H, K , our concern will focus on the Jacobson-Witt type for trying to attacking this problem as a first step in this paper.

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1.2. Let $\mathcal{A}(n) = \text{Span}_{\mathbb{F}}\{x^{(\alpha)} \mid \alpha \in \mathbb{Z}_+^n\}$ be the divided power algebra. Its multiplication is defined by

$$x^{(\alpha)}x^{(\beta)} = \binom{\alpha + \beta}{\alpha} x^{(\alpha + \beta)},$$

where $\binom{\alpha + \beta}{\alpha} = \prod_{i=1}^n \binom{\alpha_i + \beta_i}{\alpha_i}$, $\binom{\alpha_i + \beta_i}{\alpha_i} = \frac{(\alpha_i + \beta_i)!}{\alpha_i! \beta_i!}$.

The Jacobson-Witt algebra $\mathcal{W}(n)$ is defined to be the derivation algebra of divided power algebra $\mathcal{A}(n)$ (see [12] for more details):

$$\mathcal{W}(n) = \text{Der} \mathcal{A}(n) = \left\{ \sum_{i=1}^n f_i \partial_i \mid f_i \in \mathcal{A}(n) \right\}.$$

Set $\deg(x^{(\alpha)}) = |\alpha| = \sum_{i=1}^n \alpha_i$, $\deg(\partial_i) = -1$. Then there exists a natural gradation over $\mathcal{W}(n)$:

$$\mathcal{W}(n) = \bigoplus_{i \geq -1} \mathcal{W}(n)_i = \bigoplus_{i \geq -1} \text{Span}_{\mathbb{F}} \left\{ x^{(\alpha)} \partial_i \mid |\alpha| = i + 1 \right\},$$

where $\mathcal{W}(n)_i$ is the subspace of the derivations of degree i .

Note that $\mathcal{W}(n)_0 \cong gl_n = sl_n \oplus \mathbb{C}c$, where $c = \text{diag}(1, \dots, 1)$ and the isomorphism is obtained by mapping the differential operators $x^{(\varepsilon_i)} \partial_j$'s to the matrix units E_{ij} 's, where the action of the operator $x^{(\varepsilon_i)} \partial_j$ acting on $x^{(\alpha)}$ is given by $x^{(\varepsilon_i)} \partial_j(x^{(\alpha)}) = \binom{\alpha + \varepsilon_i - \varepsilon_j}{\varepsilon_i} x^{(\alpha + \varepsilon_i - \varepsilon_j)}$. This means that gl_n is realized as certain differential operators defined over the divided power algebra $\mathcal{A}(n)$.

1.3. It is natural to ask how to realize $U_q(gl_n)$ using quantum differential operators such that $U_q(\mathcal{W}(n)_0) \cong U_q(gl_n)$? To deal with this problem, the first author [4] introduced the quantum version of $\mathcal{A}(n)$, namely, the quantum divided power algebra $\mathcal{A}_q(n)$. As a vector space, it is generated by $\{x^{(\alpha)} \mid \alpha \in \mathbb{Z}_+^n\}$ and its multiplication is given by:

$$x^{(\alpha)}x^{(\beta)} = q^{\alpha * \beta} \left[\begin{matrix} \alpha + \beta \\ \alpha \end{matrix} \right] x^{(\alpha + \beta)},$$

where $\left[\begin{matrix} \alpha + \beta \\ \alpha \end{matrix} \right] = \prod_{i=1}^n \left[\begin{matrix} \alpha_i + \beta_i \\ \alpha_i \end{matrix} \right]$, $\left[\begin{matrix} \alpha_i + \beta_i \\ \alpha_i \end{matrix} \right] = \frac{[\alpha_i + \beta_i]!}{[\alpha_i]! [\beta_i]!}$, $[\alpha_i]! = [\alpha_i][\alpha_i - 1] \cdots [1]$, $[\alpha_i] = \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}}$, $(\alpha_i, \beta_i \in \mathbb{Z}_+)$ and for the definition of $\alpha * \beta$, see [4] (also see subsection 1.7). We will write $x^{(\varepsilon_i)}$ simply x_i when no confusion arises.

Also, the first author [4] introduced the quantum differential operators and algebra automorphisms defined over $\mathcal{A}_q(n)$ as follows.

For $1 \leq i \leq n$, $\frac{\partial_q}{\partial x_i}$ is defined as the special q -derivatives over $\mathcal{A}_q(n)$ by

$$(1.1) \quad \frac{\partial_q}{\partial x_i}(x^{(\beta)}) = q^{-\varepsilon_i * \beta}(x^{(\beta - \varepsilon_i)}), \quad \forall x^{(\beta)} \in \mathcal{A}_q(n).$$

For simplicity of notation, we write ∂_i for $\frac{\partial q}{\partial x_i}$.

For $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}_+^n$, define the algebra automorphisms σ_i and $\Theta(\alpha)$ of quantum divided power algebra as follows:

$$(1.2) \quad \sigma_i(x^{(\beta)}) = q^{\beta_i}(x^{(\beta)}), \quad \forall x^{(\beta)} \in \mathcal{A}_q,$$

$$(1.3) \quad \Theta(\alpha)(x^{(\beta)}) = \theta(\alpha, \beta)(x^{(\beta)}), \quad \forall x^{(\beta)} \in \mathcal{A}_q.$$

Taking $q = 1$, we have $\sigma_i = \text{id} = \Theta(\alpha)$.

These quantum differential operators and automorphisms yield a Hopf algebra D_q (see subsection 1.8 for more details) such that the quantum divided power algebra \mathcal{A}_q is a D_q -module algebra in the sense of [13]. This allows us to make their smash product algebra $\mathcal{A}_q \# D_q$ and define it to be the quantum Weyl algebra (see [4]), denoted by $\mathcal{W}_q(2n)$, which is different from those that have appeared in the literature (for instance [2], [3], [11], etc.).

Although $\mathcal{W}_q(2n)$ itself is not a Hopf algebra, it contains many interesting Hopf subalgebras such as $U_q(gl_n)$ and $U_q(sl_n)$ (these can be realized as some quantum differential operators in $W_q(2n)$), and also provides some strong smash product algebras in rank 1 case (revised version for root of unity) for calculating their cyclic homologies (see [15]).

Proposition 1.1. ([4, Thm. 4.1]) *For any monomial $x^{(\beta)} \in \mathcal{A}_q(n)$ and $1 \leq i \leq n-1$, set*

$$\begin{aligned} e_i(x^{(\beta)}) &= x_i \partial_{i+1} \sigma_i(x^{(\beta)}), \\ f_i(x^{(\beta)}) &= \sigma_i^{-1} x_{i+1} \partial_i(x^{(\beta)}), \\ K_i(x^{(\beta)}) &= \sigma_i \sigma_{i+1}^{-1}(x^{(\beta)}), \\ K_i^{-1}(x^{(\beta)}) &= \sigma_i^{-1} \sigma_{i+1}(x^{(\beta)}). \end{aligned}$$

These formulas define the structure of a $U_q(sl_n)$ -module algebra on $\mathcal{A}_q(n)$.

Corollary 1.2. ([4, Coro. 4.1]) *For any monomial $x^{(\beta)} \in \mathcal{A}_q(n)$, set*

$$\begin{aligned} e_i(x^{(\beta)}) &= x_i \partial_{i+1} \sigma_i(x^{(\beta)}) \quad (1 \leq i \leq n-1), \\ f_i(x^{(\beta)}) &= \sigma_i^{-1} x_{i+1} \partial_i(x^{(\beta)}) \quad (1 \leq i \leq n-1), \\ k_i(x^{(\beta)}) &= \sigma_i(x^{(\beta)}) \quad (1 \leq i \leq n), \\ k_i^{-1}(x^{(\beta)}) &= \sigma_i^{-1}(x^{(\beta)}) \quad (1 \leq i \leq n). \end{aligned}$$

These formulas define the structure of a $U_q(gl_n)$ -module algebra over $\mathcal{A}_q(n)$.

Actually, this realizes the $U_q(\mathcal{W}(n)_0) \cong U_q(gl_n)$ as some quantum differential operators in n variables defined over $\mathcal{A}_q(n)$.

1.4. Let $L_1 = \text{Span}_{\mathbb{R}}\{x_k \sum_{i=1}^n x_i \partial_i \mid 1 \leq k \leq n\} \subset \mathcal{W}(n)_1$ be an abelian Lie subalgebra of $\mathcal{W}(n)$. Set $L_{n+1} := \mathcal{W}(n)_{-1} \oplus \mathcal{W}(n)_0 \oplus L_1 \subset \mathcal{W}(n)$. It is easy to check that L_{n+1} is also a Lie subalgebra of $\mathcal{W}(n)$, which is isomorphic to sl_{n+1} by Zhao-Xu [16].

Lemma 1.3. ([16, Lemma 3.1.1]) *The special linear Lie algebra sl_{n+1} is isomorphism to L_{n+1} with the following identification of Chevalley generators*

$$\begin{aligned} e_i &= x_i \partial_{i+1}, \quad (1 \leq i \leq n-1), & e_n &= x_n \sum_{i=1}^n x_i \partial_i, \\ f_i &= x_{i+1} \partial_{x_i}, \quad (1 \leq i \leq n-1), & f_n &= -\partial_n, \\ h_i &= x_i \partial_i - x_{i+1} \partial_{i+1}, \quad (1 \leq i \leq n-1), & h_n &= \sum_{i=1}^n x_i \partial_{x_i} + x_n \partial_n. \end{aligned}$$

This means that one can give the differential operator realization for the higher rank special linear Lie algebra sl_{n+1} using the differential operators in n variables (rather than $n+1$ variables).

In this paper, we will generalize the above result to the quantum case, that is, give the quantum differential operators realization for the higher rank quantized enveloping algebra $U_q(sl_{n+1})$ using certain quantum differential operators in $\mathcal{W}_q(2n)$, namely, we realize the quantum algebra of Lie subalgebra $L_{n+1} := \mathcal{W}(n)_{-1} \oplus \mathcal{W}(n)_0 \oplus L_1$ of $\mathcal{W}(n)$ as certain quantum differential operators defined over quantum divided power algebra $\mathcal{A}_q(n)$. Hence we go ahead towards quantizing the entire Lie algebra $\mathcal{W}(n)$. We emphasize that we are able to realize the higher rank quantum group $U_q(sl_{n+1})$ using quantum differential operators in n variables defined over $\mathcal{A}_q(n)$ rather than $n+1$ variables defined over $\mathcal{A}_q(n+1)$, since the latter follows directly from Proposition 1.1.

1.5. In [4], after giving the realization of simple root vectors of $U_q(gl_n)$, the first author gave further the realization of all root vectors of $U_q(gl_n)$, which is proved to be coincident with one of four kinds of root vectors in the sense of Lusztig using the braid automorphisms of $U_q(sl_n)$ (also known as Lusztig symmetries [10]).

For $1 \leq i < j \leq n$, let $e_{ij} = x_i \partial_j \sigma_i$, and for $1 \leq j < i \leq n$, set $e_{ij} = \sigma_j^{-1} x_i \partial_j$.

Proposition 1.4. ([4, Prop. 4.6]) *Identifying the generators of $U_q(sl_n)$ with certain quantum differential operators in $\mathcal{W}_q(2n)$, i.e., $e_i := e_{i,i+1}, f_i := e_{i+1,i}, K_i := \sigma_i \sigma_{i+1}^{-1}$ ($1 \leq i \leq n-1$), we have*

(1) $e_{\alpha_{ij}}$ correspond to $e_{i,j}$ ($1 \leq i < j \leq n$), where $e_{\alpha_{ij}}$ are the positive root vectors associated to those positive roots $\alpha_{ij} = \varepsilon_i - \varepsilon_j$.

(2) $f_{\alpha_{ji}}$ correspond to $e_{i,j}$ ($1 \leq j < i \leq n$), where $f_{\alpha_{ji}}$ are the negative root vectors associated to those negative roots $\alpha_{ji} = \varepsilon_i - \varepsilon_j$.

In this paper, after giving the realization of simple root vectors of higher rank quantum group $U_q(sl_{n+1})$, we also realize further the non-simple root vectors using the Lusztig symmetries and show that one of the four kinds of Lusztig root vectors can be specified under the realization.

1.6. For the convenience of the reader, we summarize the relevant results without proofs to make our exposition self-contained.

The quantum group $U_q(sl_n)$ [8] is an associative algebra over \mathbb{K} generated by the elements $e_i, f_i, K_i^{\pm 1}$ ($1 \leq i \leq n-1$), subject to the relations:

$$\begin{aligned}
 (R1) \quad & K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \\
 (R2) \quad & K_i e_j K_i^{-1} = q^{a_{ij}} e_j, \quad K_i f_j K_i^{-1} = q^{-a_{ij}} f_j, \\
 (R3) \quad & [e_i, f_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
 (R4) \quad & e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \quad (|i - j| = 1), \\
 (R5) \quad & e_i e_j = e_j e_i \quad (|i - j| > 1), \\
 (R6) \quad & f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \quad (|i - j| = 1), \\
 (R7) \quad & f_i f_j = f_j f_i \quad (|i - j| > 1).
 \end{aligned}$$

where $q \in \mathbb{K}^*$ and (a_{ij}) is the Cartan matrix of type A_{n-1} .

$U_q(sl_n)$ has a Hopf algebra structure with the comultiplication, the counit and the antipode given by:

$$\begin{aligned}
 \Delta(K_i^{\pm 1}) &= K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i, \\
 \Delta(f_i) &= f_i \otimes 1 + K_i^{-1} \otimes f_i, \quad \varepsilon(K_i^{\pm 1}) = 1, \\
 \varepsilon(e_i) &= \varepsilon(f_i) = 0, \quad S(K_i^{\pm 1}) = K_i^{\mp 1}, \\
 S(e_i) &= -e_i K_i^{-1}, \quad S(f_i) = -K_i f_i.
 \end{aligned}$$

Let $P = \text{Span}_{\mathbb{Z}}\{\varepsilon_1, \dots, \varepsilon_n\}$ be the weight lattice for gl_n and $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1\}$ be the set of simple roots of gl_n . Define a symmetric bilinear form $\langle \cdot, \cdot \rangle : P \times P \rightarrow \mathbb{Z}$ such that

$$\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}.$$

Now we can give the presentation of $U_q(gl_n)$ as follows:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i \quad (1 \leq i \leq n),$$

$$K_i = k_i k_{i+1}^{-1} \quad (1 \leq i \leq n-1),$$

$$k_i e_j k_i^{-1} = q^{\langle \varepsilon_i, \alpha_j \rangle} e_j, \quad k_i f_j k_i^{-1} = q^{-\langle \varepsilon_i, \alpha_j \rangle} f_j.$$

and keep the remaining relations in the definition of $U_q(sl_n)$ invariant.

1.7. We introduce a product $*$ and a mapping θ of two integer n -tuples following [4]. For any $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$, define $\alpha * \beta := \sum_{j=1}^{n-1} \sum_{i>j} \alpha_i \beta_j$ and define $\theta : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow k^*$ by $\theta(\alpha, \beta) = q^{\alpha * \beta - \beta * \alpha}$.

Lemma 1.5. ([4, §2.1]) *We list some useful properties of the product $*$ and the mapping θ :*

$$\begin{aligned} (\alpha + \beta) * \gamma &= \alpha * \gamma + \beta * \gamma, & \alpha * (\beta + \gamma) &= \alpha * \beta + \alpha * \gamma, \\ \theta(\alpha + \beta, \gamma) &= \theta(\alpha, \gamma) \theta(\beta, \gamma), & \theta(\alpha, \beta + \gamma) &= \theta(\alpha, \beta) \theta(\alpha, \gamma), \\ \theta(\alpha, 0) &= 1 = \theta(0, \alpha), & \theta(\alpha, \beta) \theta(\beta, \alpha) &= 1 = \theta(\alpha, \alpha). \end{aligned}$$

Taking α or β to be ε_i or ε_j , we have

$$\varepsilon_i * \beta = \sum_{s < i} \beta_s, \quad \beta * \varepsilon_i = \sum_{s > i} \beta_s \quad (1 \leq i \leq n), \quad \theta(\varepsilon_i, \varepsilon_j) = \begin{cases} q & (i > j) \\ 1 & (i = j) \\ q^{-1} & (i < j) \end{cases}$$

where $\varepsilon_i = (\delta_{1i}, \dots, \delta_{ni})$, $(1 \leq i < n)$.

1.8. In this section, we will give the definition of quantum Weyl algebra $\mathcal{W}_q(2n)$ which is defined in [4] as a kind of smash product algebras.

Let us first recall some elementary properties of the quantum differential operators and algebra automorphisms in subsection 1.3, which will be needed in the construction of $\mathcal{W}_q(2n)$.

Proposition 1.6. ([4, Prop.3.1]) (1) $\Theta(\alpha)\Theta(\beta) = \Theta(\alpha + \beta)$, in particular, $\Theta(-\alpha_i) = \sigma_i \sigma_{i+1}$, where α_i is a simple root in a root system of type A_{n-1} .

(2) ∂_i is a $(\Theta(-\varepsilon_i)\sigma_i^{\pm 1}, \sigma_i^{\mp 1})$ -derivation of \mathcal{A}_q , that is to say,

$$\partial_i(x^{(\beta)}x^{(\gamma)}) = \partial_i(x^{(\beta)})\sigma_i^{\mp 1}(x^{(\gamma)}) + \Theta(-\varepsilon_i)\sigma_i^{\pm 1}(x^{(\beta)})\partial_i(x^{(\gamma)}).$$

(3) $\partial_i \partial_j = \theta(\varepsilon_i, \varepsilon_j) \partial_j \partial_i$.

(4) $x^{(\alpha)}(x^{(\beta)}x^{(\gamma)}) = (x^{(\alpha)}x^{(\beta)})x^{(\gamma)} = \theta(\alpha, \beta)x^{(\beta)}(x^{(\alpha)}x^{(\gamma)}) = \Theta(\alpha)(x^{(\beta)})(x^{(\alpha)}x^{(\gamma)})$.

(5) $\sigma_i(x^{(\beta)}x^{(\gamma)}) = \sigma_i(x^{(\beta)})\sigma_i(x^{(\gamma)})$, $\Theta(\alpha)(x^{(\beta)}x^{(\gamma)}) = \Theta(\alpha)(x^{(\beta)})\Theta(\alpha)(x^{(\gamma)})$.

(6) $x^{(\alpha)}\partial_i$ is a $(\Theta(\alpha - \varepsilon_i)\sigma_i^{\pm 1}, \sigma_i^{\mp 1})$ -derivation of $\mathcal{A}_q(n)$, namely,

$$x^{(\alpha)}\partial_i(x^{(\beta)}x^{(\gamma)}) = x^{(\alpha)}\partial_i(x^{(\beta)})\sigma_i^{\mp 1}(x^{(\gamma)}) + \Theta(\alpha - \varepsilon_i)\sigma_i^{\pm 1}(x^{(\beta)})x^{(\alpha)}\partial_i(x^{(\gamma)}).$$

Let D_q be the associative algebra over \mathbb{K} generated by the elements $\Theta(\pm\varepsilon_i), \sigma_i^{\pm 1}, \partial_i, (1 \leq i \leq n)$, satisfying the following relations:

$$\begin{aligned}\Theta(\pm\varepsilon_i)\Theta(\mp\varepsilon_i) &= 1 = \sigma_i^{\pm 1}\sigma_i^{\mp 1}, & \Theta(-\varepsilon_i+\varepsilon_{i+1}) &= \sigma_i\sigma_{i+1}, \\ \Theta(\varepsilon_i)\Theta(\varepsilon_j) &= \Theta(\varepsilon_i+\varepsilon_j) = \Theta(\varepsilon_j)\Theta(\varepsilon_i), & \sigma_i\sigma_j &= \sigma_j\sigma_i, \\ \sigma_i\Theta(\varepsilon_j) &= \Theta(\varepsilon_j)\sigma_i, & \Theta(\varepsilon_i)\partial_j\Theta(-\varepsilon_i) &= \theta(\varepsilon_j, \varepsilon_i)\partial_j, \\ \sigma_i\partial_j\sigma_i^{-1} &= q^{-\delta_{ij}}\partial_j, & \partial_i\partial_j &= \theta(\varepsilon_i, \varepsilon_j)\partial_j\partial_i.\end{aligned}$$

Furthermore, D_q can be equipped with two Hopf algebra structures with the structure mappings $\Delta^{(\pm)}, \varepsilon$ and $S^{(\pm)}$ are given by

$$\begin{aligned}\Delta^{(\pm)}(\sigma_i^{\pm 1}) &= \sigma_i^{\pm 1} \otimes \sigma_i^{\pm 1}, & \Delta^{(\pm)}(\Theta(\pm\varepsilon_i)) &= \Theta(\pm\varepsilon_i) \otimes \Theta(\pm\varepsilon_i), \\ \Delta^{(\pm)}(\partial_i) &= \partial_i \otimes \sigma_i^{\mp 1} + \Theta(-\varepsilon_i)\sigma_i^{\pm 1} \otimes \partial_i, & \varepsilon(\partial_i) &= 0, \\ \varepsilon(\sigma_i^{\pm 1}) &= 1 = \varepsilon(\Theta(\pm\varepsilon_i)), & S^{(\pm)}(\sigma_i^{\pm 1}) &= \sigma_i^{\mp 1}, \\ S^{(\pm)}(\Theta(\pm\varepsilon_i)) &= \Theta(\mp\varepsilon_i), & S^{(\pm)}(\partial_i) &= -q^{\pm 1}\Theta(\varepsilon_i)\partial_i.\end{aligned}$$

Let $D_q^{(\pm)} := (D_q, \Delta^{(\pm)}, \varepsilon, S^{(\pm)})$ denote the two Hopf algebras mentioned above. Then from Proposition 1.6, we conclude that $\mathcal{A}_q(n)$ is a left $D_q^{(\pm)}$ -module algebra, which gives rise to smash product algebras $\mathcal{A}_q(n) \# D_q^{(\pm)}$ (see [4]).

Definition 1.7. ([4, Def. 3.5]) Let $\mathcal{W}_q(2n)$ be the associative algebra over \mathbb{K} generated by the symbols $\Theta(\pm\varepsilon_i), \sigma_i^{\pm 1}, x_i, \partial_i$ ($1 \leq i \leq n$) with the following defining relations:

$$\begin{aligned}\Theta(\pm\varepsilon_i)\Theta(\mp\varepsilon_i) &= 1 = \sigma_i^{\pm 1}\sigma_i^{\mp 1}, & \Theta(-\varepsilon_i+\varepsilon_{i+1}) &= \sigma_i\sigma_{i+1}, \\ \Theta(\varepsilon_i)\Theta(\varepsilon_j) &= \Theta(\varepsilon_i+\varepsilon_j) = \Theta(\varepsilon_j)\Theta(\varepsilon_i), & \sigma_i\sigma_j &= \sigma_j\sigma_i, \\ \sigma_i\Theta(\varepsilon_j) &= \Theta(\varepsilon_j)\sigma_i, & \Theta(\varepsilon_i)x_j\Theta(-\varepsilon_i) &= \theta(\varepsilon_i, \varepsilon_j)x_j, \\ \Theta(\varepsilon_i)\partial_j\Theta(-\varepsilon_i) &= \theta(\varepsilon_j, \varepsilon_i)\partial_j, & \sigma_ix_j\sigma_i^{-1} &= q^{\delta_{ij}}x_j, \\ \sigma_i\partial_j\sigma_i^{-1} &= q^{-\delta_{ij}}\partial_j, & x_ix_j &= \theta(\varepsilon_i, \varepsilon_j)x_jx_i, \\ \partial_i\partial_j &= \theta(\varepsilon_i, \varepsilon_j)\partial_j\partial_i, & \partial_ix_j &= \theta(\varepsilon_j, \varepsilon_i)x_j\partial_i \quad (i \neq j), & \partial_ix_i - q^{\pm 1}x_i\partial_i &= \sigma_i^{\mp 1},\end{aligned}$$

where the last relations are equivalent to the relations:

$$\partial_ix_i = \frac{q\sigma_i - (q\sigma_i)^{-1}}{q - q^{-1}}, \quad x_i\partial_i = \frac{\sigma_i - \sigma_i^{-1}}{q - q^{-1}}.$$

2. REALIZATION OF THE SIMPLE ROOT VECTORS

2.1. Based on the results above, we will realize the higher rank quantized enveloping algebra $U_q(sl_{n+1})$ as some quantum differential operators in $\mathcal{W}_q(2n)$ defined over the quantum divided algebra $\mathcal{A}_q(n)$ such that $\mathcal{A}_q(n)$ is made into a nonhomogeneous $U_q(sl_{n+1})$ -module.

This amounts to saying that we can quantize the components of degree -1 and degree 0 of $\mathcal{W}(n)$, together with a n -dimensional subspace L_1 of $\mathcal{W}(n)_1$.

First of all, we give the following lemma.

Lemma 2.1. $\sum_{k=1}^n \theta(\varepsilon_k, \beta)[\beta_k] = \sum_{k=1}^n \theta(\varepsilon_k, \beta + m(\varepsilon_i - \varepsilon_{i+1}))[(\beta + m(\varepsilon_i - \varepsilon_{i+1}))_k], \forall m \in \mathbb{Z}.$

Proof. It suffices to verify

$$\begin{aligned} \theta(\varepsilon_i, \beta)[\beta_i] + \theta(\varepsilon_{i+1}, \beta)[\beta_{i+1}] &= \theta(\varepsilon_i, \beta + m(\varepsilon_i - \varepsilon_{i+1}))[(\beta + m(\varepsilon_i - \varepsilon_{i+1}))_i] \\ &\quad + \theta(\varepsilon_{i+1}, \beta + m(\varepsilon_i - \varepsilon_{i+1}))[(\beta + m(\varepsilon_i - \varepsilon_{i+1}))_{i+1}]. \end{aligned}$$

$$\begin{aligned} RHS &= \theta(\varepsilon_i, \beta)\theta(\varepsilon_i, \varepsilon_i)^m\theta(\varepsilon_i, -\varepsilon_{i+1})^m[(\beta + m(\varepsilon_i - \varepsilon_{i+1}))_i] \\ &\quad + \theta(\varepsilon_{i+1}, \beta)\theta(\varepsilon_{i+1}, \varepsilon_i)^m\theta(\varepsilon_{i+1}, -\varepsilon_{i+1})^m[(\beta + m(\varepsilon_i - \varepsilon_{i+1}))_{i+1}] \\ &= q^m\theta(\varepsilon_i, \beta)[\beta_i + m] + q^m\theta(\varepsilon_{i+1}, \beta)[\beta_{i+1} - m] \\ &= \theta(\varepsilon_i, \beta)[\beta_i] + \theta(\varepsilon_{i+1}, \beta)[\beta_{i+1}] + (\theta(\varepsilon_i, \beta)q^{\beta_i} - \theta(\varepsilon_{i+1}, \beta)q^{-\beta_{i+1}})q^m[m] \\ &= LHS. \end{aligned}$$

This confirms the identity. □

2.2. We are now in a position to state and prove one of our main results.

Theorem 2.2. *For any monomial $x^{(\beta)} \in \mathcal{A}_q(n)$ and $1 \leq i \leq n$, set*

$$\begin{aligned} e_i(x^{(\beta)}) &= x_i \partial_{i+1} \sigma_i(x^{(\beta)}), \\ f_i(x^{(\beta)}) &= \sigma_i^{-1} x_{i+1} \partial_i(x^{(\beta)}), \\ K_i(x^{(\beta)}) &= \sigma_i \sigma_{i+1}^{-1}(x^{(\beta)}), \\ e_n(x^{(\beta)}) &= \left(\prod_{i=1}^{n-1} \sigma_i^{-1} \right) \left(x_n \sum_{i=1}^n x_i \partial_i \Theta(\varepsilon_i) \right) (x^{(\beta)}), \\ f_n(x^{(\beta)}) &= \left(-\partial_n \prod_{i=1}^{n-1} \sigma_i \right) (x^{(\beta)}), \\ K_n(x^{(\beta)}) &= \left(\sigma_n \prod_{i=1}^n \sigma_i \right) (x^{(\beta)}). \end{aligned}$$

These formulas make $\mathcal{A}_q(n)$ into a nonhomogeneous $U_q(sl_{n+1})$ -module.

Remark 2.3. Taking $q = 1$, we recover the realization of sl_{n+1} (cf. Lemma 1.3).

Proof. By Proposition 1.1, we only need to check the algebraic relations that the generators $\{e_n, f_n, K_n^{\pm 1}\}$ occur.

Using Lemma 1.5 and formulas (1.3), (1.1) and (2.1), we obtain

$$\begin{aligned}
e_i(x^{(\beta)}) &= q^{\beta_i - \varepsilon_{i+1} * \beta + \varepsilon_i * (\beta - \varepsilon_{i+1})} \begin{bmatrix} \beta + \varepsilon_i - \varepsilon_{i+1} \\ \varepsilon_i \end{bmatrix} x^{(\beta + \varepsilon_i - \varepsilon_{i+1})} \\
&= [\beta_i + 1] x^{(\beta + \varepsilon_i - \varepsilon_{i+1})}, \\
f_i(x^{(\beta)}) &= q^{-\varepsilon_i * \beta + \varepsilon_{i+1} * (\beta - \varepsilon_i) - (\beta_i - 1)} \begin{bmatrix} \beta - \varepsilon_i + \varepsilon_{i+1} \\ \varepsilon_{i+1} \end{bmatrix} x^{(\beta - \varepsilon_i + \varepsilon_{i+1})} \\
&= [\beta_{i+1} + 1] x^{(\beta - \varepsilon_i + \varepsilon_{i+1})}, \\
e_n(x^{(\beta)}) &= [\beta_n + 1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) x^{(\beta + \varepsilon_n)}, \\
f_n(x^{(\beta)}) &= -x^{(\beta - \varepsilon_n)}, \quad K_n(x^{(\beta)}) = q^{\sum_{i=1}^n \beta_i + \beta_n} x^{(\beta)}.
\end{aligned}$$

First of all, it is clear that (R1) holds.

For (R2), we have

$$\begin{aligned}
K_i e_n K_i^{-1}(x^{(\beta)}) &= q^{\beta_{i+1} - \beta_i} [\beta_{n+1}] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) K_i x^{(\beta + \varepsilon_n)} \\
&= q^{\beta_{i+1} - \beta_i} q^{\beta_i - \beta_{i+1} + \delta_{i,n} - \delta_{i+1,n}} e_n(x^{(\beta)}) \\
&= q^{\delta_{i,n} - \delta_{i+1,n}} e_n(x^{(\beta)}) \\
&= q^{a_{i,n}} e_n(x^{(\beta)}),
\end{aligned}$$

for $1 \leq i \leq n-1$.

$$\begin{aligned}
K_n e_n K_n^{-1}(x^{(\beta)}) &= q^{-\left(\sum_{i=1}^n \beta_i + \beta_n\right)} K_n e_n(x^{(\beta)}) \\
&= q^{-\left(\sum_{i=1}^n \beta_i + \beta_n\right)} [\beta_{n+1}] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) K_n x^{(\beta + \varepsilon_n)} \\
&= q^{-\left(\sum_{i=1}^n \beta_i + \beta_n\right)} q^{\left(\sum_{i=1}^n \beta_i + \beta_n\right) + 2} e_n(x^{(\beta)}) \\
&= q^{a_{n,n}} e_n(x^{(\beta)}).
\end{aligned}$$

Similarly, for $1 \leq i \leq n-1$, we obtain

$$\begin{aligned}
K_i f_n K_i^{-1}(x^{(\beta)}) &= -q^{\beta_{i+1} - \beta_i} q^{-\beta_{i+1} + \beta_i + \delta_{i+1,n}} x^{(\beta - \varepsilon_n)} \\
&= q^{\delta_{i+1,n}} f_n(x^{(\beta)}) = q^{-a_{i,n}} f_n(x^{(\beta)}).
\end{aligned}$$

For $i = n$, we have

$$K_n f_n K_n^{-1}(x^{(\beta)}) = -q^{-\left(\sum_{i=1}^n \beta_i + \beta_n\right)} q^{\left(\sum_{i=1}^n \beta_i + \beta_n\right) - 2} (x^{(\beta)}) = q^{-a_n} f_n(x^{(\beta)}).$$

As for (R3), firstly, let us prove $[e_n, f_i] = 0$, for $1 \leq i \leq n-1$. In fact, we have

$$\begin{aligned} [e_n, f_i](x^{(\beta)}) &= e_n f_i(x^{(\beta)}) - f_i e_n(x^{(\beta)}) \\ &= [\beta_{i+1} + 1][\beta_n + \delta_{i+1, n} + 1] \left(\sum_{k=1}^n \theta(\varepsilon_k, \beta - \varepsilon_i + \varepsilon_{i+1}) [(\beta - \varepsilon_i + \varepsilon_{i+1})_k] \right) x^{(\beta - \varepsilon_i + \varepsilon_{i+1} + \varepsilon_n)} \\ &\quad - [\beta_n + 1][\beta_{i+1} + \delta_{i+1, n} + 1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) x^{(\beta - \varepsilon_i + \varepsilon_{i+1} + \varepsilon_n)} = 0, \end{aligned}$$

where the last equality is a consequence of Lemma 2.1.

While for the relations $[e_i, f_n] = 0$, ($1 \leq i \leq n-1$), these follows from

$$\begin{aligned} [e_i, f_n](x^{(\beta)}) &= e_i f_n(x^{(\beta)}) - f_n e_i(x^{(\beta)}) \\ &= -[\beta_i + 1] x^{(\beta + \varepsilon_i - \varepsilon_{i+1} - \varepsilon_n)} - (-1)[\beta_i + 1] x^{(\beta + \varepsilon_i - \varepsilon_{i+1} - \varepsilon_n)} = 0. \end{aligned}$$

Finally, we shall check that $[e_n, f_n] = \frac{K_n - K_n^{-1}}{q - q^{-1}}$. Indeed,

$$\begin{aligned} [e_n, f_n](x^{(\beta)}) &= e_n f_n(x^{(\beta)}) - f_n e_n(x^{(\beta)}) \\ &= -[\beta_n] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta - \varepsilon_n) [(\beta - \varepsilon_n)_i] \right) x^{(\beta)} - (-1)[\beta_n + 1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) x^{(\beta)} \\ &= - \left(\sum_{i=1}^{n-1} q \theta(\varepsilon_i, \beta) [\beta_i] [\beta_n] + \theta(\varepsilon_n, \beta) [\beta_n - 1] [\beta_n] \right) x^{(\beta)} \\ &\quad + \left(\sum_{i=1}^{n-1} \theta(\varepsilon_i, \beta) [\beta_i] [\beta_n + 1] + \theta(\varepsilon_n, \beta) [\beta_n] [\beta_n + 1] \right) x^{(\beta)} \\ &= \left(\sum_{i=1}^n q^{-\beta_n} \theta(\varepsilon_i, \beta) [\beta_i] + q^{\beta_n} \theta(\varepsilon_n, \beta) [\beta_n] \right) x^{(\beta)} \\ &= \frac{x^{(\beta)}}{q - q^{-1}} \left(\sum_{i=1}^n q^{-\beta_n} \theta(\varepsilon_i, \beta) q^{\beta_i} + q^{\beta_n} \theta(\varepsilon_n, \beta) q^{\beta_n} - \sum_{i=1}^n q^{-\beta_n} \theta(\varepsilon_i, \beta) q^{-\beta_i} - q^{\beta_n} \theta(\varepsilon_n, \beta) q^{-\beta_n} \right) \\ &= \frac{1}{q - q^{-1}} \left(q^{\sum_{i=1}^n \beta_i + \beta_n} - q^{-\sum_{i=1}^n \beta_i - \beta_n} \right) x^{(\beta)} = \frac{K_n - K_n^{-1}}{q - q^{-1}} x^{(\beta)}. \end{aligned}$$

As for (R4), it is easily seen that

$$e_{n-1}^2 e_n(x^{(\beta)}) = [\beta_n + 1][\beta_{n-1} + 1][\beta_{n-1} + 2] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) x^{(\beta + 2\varepsilon_{n-1} - \varepsilon_n)},$$

$$e_{n-1}e_n e_{n-1}(x^{(\beta)}) = [\beta_n][\beta_{n-1}+1][\beta_{n-1}+2] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta)[\beta_i] \right) x^{(\beta+2\varepsilon_{n-1}-\varepsilon_n)},$$

$$e_n e_{n-1}^2(x^{(\beta)}) = [\beta_n-1][\beta_{n-1}+1][\beta_{n-1}+2] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta)[\beta_i] \right) x^{(\beta+2\varepsilon_{n-1}-\varepsilon_n)},$$

where the last equality follows from Lemma 2.1 when taking $m = 2$ and $i = n-1$. By definition, $[m+1] - (q+q^{-1})[m] + [m-1] = 0$, we conclude that

$$e_{n-1}^2 e_n - (q+q^{-1})e_{n-1}e_n e_{n-1} + e_n e_{n-1}^2 = 0.$$

Similarly, we have that

$$e_n^2 e_{n-1} - (q+q^{-1})e_n e_{n-1} e_n + e_{n-1} e_n^2 = 0.$$

We are now in a position to show (R5), namely, $e_i e_n = e_n e_i$ for $1 \leq i \leq n-2$.

$$\begin{aligned} e_i e_n(x^{(\beta)}) &= [\beta_n+1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta)[\beta_i] \right) e_i x^{(\beta+\varepsilon_n)} \\ &= [\beta_n+1][\beta_i+1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta)[\beta_i] \right) x^{(\beta+\varepsilon_n+\varepsilon_i-\varepsilon_{i+1})} \\ &= [\beta_n+1][\beta_i+1] \left(\sum_{k=1}^n \theta(\varepsilon_k, \beta+\varepsilon_i-\varepsilon_{i+1})[(\beta+\varepsilon_i-\varepsilon_{i+1})_k] \right) x^{(\beta+\varepsilon_n+\varepsilon_i-\varepsilon_{i+1})} \\ &= [\beta_i+1] e_n(x^{(\beta+\varepsilon_i-\varepsilon_{i+1})}) \\ &= e_n e_i(x^{(\beta)}), \end{aligned}$$

where the third equality is a consequence of Lemma 2.1.

We will prove the relation (R6) as follows. Firstly, it is not difficult to check that

$$\begin{aligned} f_{n-1}^2 f_n(x^{(\beta)}) &= -[\beta_n][\beta_n+1] x^{(\beta-2\varepsilon_{n-1}+\varepsilon_n)}, \\ f_{n-1} f_n f_{n-1}(x^{(\beta)}) &= -[\beta_n+1][\beta_n+1] x^{(\beta-2\varepsilon_{n-1}+\varepsilon_n)}, \\ f_n f_{n-1}^2(x^{(\beta)}) &= -[\beta_n+2][\beta_n+1] x^{(\beta-2\varepsilon_{n-1}+\varepsilon_n)}. \end{aligned}$$

It is easy to see that $f_{n-1}^2 f_n - (q+q^{-1})f_{n-1} f_n f_{n-1} + f_n f_{n-1}^2 = 0$. Likewise, we can show that $f_n^2 f_{n-1} - (q+q^{-1})f_n f_{n-1} f_n + f_{n-1} f_n^2 = 0$.

Finally, we check the relation (R7) as follows.

$$f_i f_n(x^{(\beta)}) = -[\beta_{i+1}+1] x^{(\beta-\varepsilon_n-\varepsilon_i+\varepsilon_{i+1})} = f_n f_i(x^{(\beta)}),$$

for $1 \leq i \leq n-2$.

This completes the proof. \square

3. REALIZATION OF THE NON-SIMPLE ROOT VECTORS

In this section, based on the realization model of the simple root vectors, we will use Lusztig symmetries to give the quantum differential operators realization for the non-simple quantum root vectors of $U_q(sl_{n+1})$. One will see that the expression of quantum root vectors by our quantum differential operators is well-matched with the action of Lusztig symmetries. This implies further that our construction is intrinsically natural.

3.1. Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ ($1 \leq i \leq n-1$) be the simple roots of sl_n and s_i ($1 \leq i \leq n-1$) be the corresponding reflections of the Weyl group $W = S_n$. Let T_i be the braid automorphism of $U_q(sl_n)$ determined by s_i introduced as $T''_{i,-1}$ by Lusztig in [10, §37.1.3]. They are defined as follows:

$$(3.1) \quad T_i(K_\mu) = K_{s_i(\mu)}, \quad T_i(e_i) = -f_i K_i^{-1}, \quad T_i(f_i) = -K_i e_i,$$

$$(3.2) \quad T_i(e_j) = e_j, \quad T_i(f_j) = f_j, \quad (|i - j| > 1),$$

$$(3.3) \quad T_i(e_j) = e_i e_j - q e_j e_i, \quad T_i(f_j) = f_j f_i - q^{-1} f_i f_j, \quad (|i - j| = 1),$$

where e_i, f_i are the simple root vectors of $U_q(sl_n)$ associated to α_i and $-\alpha_i$.

Take a reduced presentation of the longest element $w_0 = s_{i_1} s_{i_2} \cdots s_{i_m}$ in W , then

$$(3.4) \quad e_\alpha = T_{i_1} \cdots T_{i_{p-1}}(e_{i_p}), \quad f_\alpha = T_{i_1} \cdots T_{i_{p-1}}(f_{i_p})$$

are called quantum root vectors of $U_q(sl_n)$ corresponding to $\pm\alpha = \pm s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p})$, $1 \leq p \leq m$. From now on, let us fix a reduced presentation of w_0 as follows

$$w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_{n-1} s_{n-2} \cdots s_2 s_1$$

which gives a convex ordering of the positive roots system of sl_n

$$\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{14}, \alpha_{24}, \alpha_{34}, \cdots, \alpha_{1n}, \alpha_{2n}, \cdots, \alpha_{n-1,n}.$$

Hence, all the quantum root vectors of $U_q(sl_n)$ associated to this ordering are determined. For instance,

$$(3.5) \quad e_{\alpha_{1n}} = T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-2} \cdots T_2 T_1(e_{n-1}).$$

Later on, we will use the quantum root vectors of $U_q(sl_n)$ associated to this ordering.

Lemma 3.1. ([10, §39.2.4]) *The braid automorphisms T_i 's satisfy the relations*

$$T_i T_j T_i = T_j T_i T_j, \quad |i - j| = 1,$$

$$T_i T_j(e_i) = e_j, \quad |i - j| = 1,$$

$$T_i(e_j) = e_j, \quad |i - j| > 1.$$

3.2. Now let us introduce some new q -differential operators in $\mathcal{W}_q(2n)$.

Set

$$(3.6) \quad e_{s,n+1} = \left(\prod_{i \neq s} \sigma_i^{-1} \right) \left(x_s \sum_{i=1}^n x_i \partial_i \Theta(\varepsilon_i) \right), \quad 1 \leq s \leq n,$$

$$(3.7) \quad e_{n+1,s} = -\partial_s \left(\prod_{i \neq s} \sigma_i \right), \quad 1 \leq s \leq n.$$

Then we have

$$(3.8) \quad \begin{aligned} e_{s,n+1}(x^{(\beta)}) &= \left(\prod_{i \neq s} \sigma_i^{-1} \right) \left(x_s \sum_{i=1}^n x_i \partial_i \Theta(\varepsilon_i) \right) (x^{(\beta)}) \\ &= \left(\prod_{i \neq s} \sigma_i^{-1} \right) x_s \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) (x^{(\beta)}) \\ &= q^{-\sum_{i=s+1}^n \beta_i} [\beta_s + 1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) (x^{(\beta + \varepsilon_s)}), \end{aligned}$$

$$(3.9) \quad e_{n+1,s}(x^{(\beta)}) = -\partial_s \left(\prod_{i \neq s} \sigma_i \right) (x^{(\beta)}) = -q^{\sum_{i=s+1}^n \beta_i} (x^{(\beta - \varepsilon_s)}).$$

In the remainder of this section, we will show that the q -differential operators defined above are coincident with the one of four kinds of Lusztig's quantum root vectors.

It is not difficult to check that

$$(3.10) \quad e_{i,j}(x^{(\beta)}) = q^{-\sum_{i < s < j} \beta_s} [\beta_i + 1] x^{(\beta + \varepsilon_i - \varepsilon_j)}, \quad (1 \leq i < j \leq n),$$

$$(3.11) \quad e_{i,j}(x^{(\beta)}) = q^{\sum_{j < s < i} \beta_s} [\beta_i + 1] x^{(\beta - \varepsilon_j + \varepsilon_i)}, \quad (1 \leq j < i \leq n).$$

Proposition 3.2. *For any $1 \leq s < n$, $s < j \leq n$, one has*

- (1) $e_{s,n+1} = [e_{s,s+1}, e_{s+1,n+1}]_q := e_{s,s+1} e_{s+1,n+1} - q e_{s+1,n+1} e_{s,s+1},$
- (2) $e_{s,n+1} = [e_{s,j}, e_{j,n+1}]_q := e_{s,j} e_{j,n+1} - q e_{j,n+1} e_{s,j},$
- (3) $e_{n+1,s} = [e_{n+1,j}, e_{j,s}]_{q^{-1}},$
- (4) $[e_{s,n+1}, e_{n+1,s}] = \frac{\left(\prod_{i=1}^n \sigma_i \right) \sigma_s - \left(\prod_{i=1}^n \sigma_i^{-1} \right) \sigma_s^{-1}}{q - q^{-1}}, \quad \text{where } \left(\prod_{i=1}^n \sigma_i \right) \sigma_s = K_{\varepsilon_s - \varepsilon_{n+1}}.$

This proposition asserts that the expressions of $e_{s,n+1}$ and $e_{n+1,s}$ by the q -brackets are independent of the choice of j ($s < j \leq n$).

Proof. (1) For $1 \leq s < n$, using formulas (3.8) and (3.10), we have

$$\begin{aligned}
& (e_{s,s+1}e_{s+1,n+1} - qe_{s+1,n+1}e_{s,s+1})(x^{(\beta)}) \\
&= q^{-\sum_{i=s+2}^n \beta_i} [\beta_{s+1}+1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) e_{s,s+1}(x^{(\beta+\varepsilon_{s+1})}) \\
&\quad - q[\beta_s+1]e_{s+1,n+1}(x^{(\beta+\varepsilon_s-\varepsilon_{s+1})}) \\
&= q^{-\sum_{i=s+2}^n \beta_i} [\beta_{s+1}+1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) [\beta_s+1](x^{(\beta+\varepsilon_s)}) \\
&\quad - q[\beta_s+1]q^{-\sum_{i=s+2}^n \beta_i} [\beta_{s+1}] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta+\varepsilon_s-\varepsilon_{s+1}) [(\beta+\varepsilon_s-\varepsilon_{s+1})_i] \right) (x^{(\beta+\varepsilon_s)}) \\
&= q^{-\sum_{i=s+1}^n \beta_i} [\beta_s+1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) (x^{(\beta+\varepsilon_s)}) \\
&= e_{s,n+1}(x^{(\beta)}).
\end{aligned}$$

(2) We will prove (2) based on (1) and by induction.

From (1), we know that the formula holds for $s = n-1$.

Assume that $e_{j+1,n+1} = [e_{j+1,k}, e_{k,n+1}]_q$ hold for any $j+1$ with $j+1 < k < n+1$. We will prove it for j as follows.

$$\begin{aligned}
e_{j,n+1} &= [e_{j,j+1}, e_{j+1,n+1}]_q = e_{j,j+1}e_{j+1,n+1} - qe_{j+1,n+1}e_{j,j+1} \\
&= e_{j,j+1}(e_{j+1,k}e_{k,n+1} - qe_{k,n+1}e_{j+1,k}) - q(e_{j+1,k}e_{k,n+1} - qe_{k,n+1}e_{j+1,k})e_{j,j+1} \\
&= (e_{j,j+1}e_{j+1,k} - qe_{j+1,k}e_{j,j+1})e_{k,n+1} - qe_{k,n+1}(e_{j,j+1}e_{j+1,k} - qe_{j+1,k}e_{j,j+1}) \\
&= e_{j,k}e_{k,n+1} - qe_{k,n+1}e_{j,k} \\
&= [e_{j,k}, e_{k,n+1}]_q.
\end{aligned}$$

(3) For any $1 \leq s < n, s < j \leq n$, it follows from formulas (3.9) and (3.11) that

$$\begin{aligned}
& (e_{n+1,j}e_{j,s} - q^{-1}e_{j,s}e_{n+1,j})(x^{(\beta)}) \\
&= [\beta_j+1]q^{\sum_{i>s} \beta_i} e_{n+1,j}(x^{(\beta-\varepsilon_s+\varepsilon_j)}) + q^{-1} \cdot q^{\sum_{i=j+1}^n \beta_i} e_{j,s}(x^{(\beta-\varepsilon_j)}) \\
&= -[\beta_j+1]q^{\sum_{i>s} \beta_i} q^{\sum_{i=j+1}^n \beta_i} (x^{(\beta-\varepsilon_s)}) + q^{\sum_{i=j+1}^n \beta_i-1} [\beta_j]q^{\sum_{i>s} \beta_i} (x^{(\beta-\varepsilon_s)}) \\
&= -q^{\sum_{i=s+1}^n \beta_i} (x^{(\beta-\varepsilon_s)}) = e_{n+1,s}(x^{(\beta)}).
\end{aligned}$$

(4) For any $1 \leq s < n$, applying formulas (3.8) and (3.9), we obtain

$$(e_{s,n+1}e_{n+1,s} - e_{n+1,s}e_{s,n+1})(x^{(\beta)})$$

$$\begin{aligned}
&= -q^{\sum_{i=s+1}^n \beta_i} e_{s,n+1}(x^{(\beta-\varepsilon_s)}) - q^{-\sum_{i=s+1}^n \beta_i} [\beta_s+1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) e_{n+1,s}(x^{(\beta+\varepsilon_s)}) \\
&= -[\beta_s] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta-\varepsilon_s) [(\beta-\varepsilon_s)_i] \right) (x^{(\beta)}) + [\beta_s+1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) (x^{(\beta)}) \\
&= -[\beta_s] \left(\sum_{i=1}^{s-1} \theta(\varepsilon_i, \beta-\varepsilon_s) [\beta_i] + \theta(\varepsilon_s, \beta) [\beta_s-1] + \sum_{i=s+1}^n \theta(\varepsilon_i, \beta-\varepsilon_s) [\beta_i] \right) (x^{(\beta)}) \\
&\quad + [\beta_s+1] \left(\sum_{i=1}^{s-1} \theta(\varepsilon_i, \beta) [\beta_i] + \theta(\varepsilon_s, \beta) [\beta_s] + \sum_{i=s+1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) (x^{(\beta)}) \\
&= \sum_{i=1}^{s-1} \left(([\beta_s]q + q^{-\beta_s}) \theta(\varepsilon_i, \beta) [\beta_i] - [\beta_s]q \theta(\varepsilon_i, \beta) [\beta_i] \right) (x^{(\beta)}) \\
&\quad + \left(([\beta_s]q^{-1} + q^{\beta_s}) \theta(\varepsilon_s, \beta) [\beta_s] - [\beta_s] \theta(\varepsilon_s, \beta) ([\beta_s]q^{-1} - q^{-\beta_s}) \right) (x^{(\beta)}) \\
&\quad + \sum_{i=s+1}^n \left(([\beta_s]q^{-1} + q^{\beta_s}) \theta(\varepsilon_i, \beta) [\beta_i] - [\beta_s]q^{-1} \theta(\varepsilon_i, \beta) [\beta_i] \right) (x^{(\beta)}) \\
&= \left(q^{-\sum_{i=s}^n \beta_i} \left[\sum_{i=1}^s \beta_i \right] + q^{\sum_{i=1}^s \beta_i} \left[\sum_{i=s}^n \beta_i \right] \right) (x^{(\beta)}) = \left[\beta_s + \sum_{i=1}^n \beta_i \right] (x^{(\beta)}) \\
&= \frac{(\prod_{i=1}^n \sigma_i) \sigma_s - (\prod_{i=1}^n \sigma_i^{-1}) \sigma_s^{-1}}{q - q^{-1}} (x^{(\beta)}).
\end{aligned}$$

Thus, we complete the proof. \square

3.3. We recall that $e_{ij} = x_i \partial_j \sigma_i$ ($1 \leq i < j \leq n$), $e_{ij} = \sigma_j^{-1} x_i \partial_j$ ($1 \leq j < i \leq n$) as given in the Introduction and $e_{s,n+1}, e_{n+1,s}$ are defined by formulas (3.6) and (3.7).

Theorem 3.3. *If we identify the generators of $U_q(sl_{n+1})$ with certain q -differential operators in $\mathcal{W}_q(2n)$, namely, $e_i := e_{i,i+1}$, $f_i := e_{i+1,i}$, $K_i := \sigma_i \sigma_{i+1}^{-1}$, $1 \leq i < n$, and $e_n := e_{n,n+1}$, $f_n := e_{n+1,n}$, $K_n := \sigma_n \prod_{i=1}^n \sigma_i$, then we have*

(1) $e_{\alpha_{ij}}$ correspond to $e_{i,j}$ ($1 \leq i < j \leq n+1$), where $e_{\alpha_{ij}}$ are positive root vectors associated to $\alpha_{ij} = \varepsilon_i - \varepsilon_j$.

(2) $f_{\alpha_{ji}}$ correspond to $e_{i,j}$ ($1 \leq j < i \leq n+1$), where $f_{\alpha_{ji}}$ are negative root vectors associated to $\alpha_{ji} = \varepsilon_i - \varepsilon_j$.

In order to prove the theorem, we first establish an auxiliary lemma.

Lemma 3.4. *With the identification as in Theorem 3.3, one has*

(1) *If $2 \leq s+1 < j \leq n+1$, then $[e_{sj}, T_s(e_s)]_q = e_{s+1,j}$.*

(2) If $2 \leq s+1 < j \leq n+1$, then $[T_s(f_s), e_{js}]_{q^{-1}} = e_{j,s+1}$.

Proof. (1) From [4, Lemma 4.6], it is clear that the formula holds for $2 \leq s+1 < j \leq n$. Hence it suffices to show the case $j = n+1$. Using formulas (3.1), (3.8) and (3.11), we have

$$\begin{aligned}
LHS(x^{(\beta)}) &= (q f_s K_s^{-1} e_{s,n+1} - e_{s,n+1} f_s K_s^{-1})(x^{(\beta)}) \\
&= q^{-\sum_{i=s+1}^n \beta_i + 1} [\beta_s + 1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) f_s K_s^{-1} (x^{(\beta + \varepsilon_s)}) \\
&\quad - q^{-\beta_s + \beta_{s+1}} [\beta_{s+1} + 1] e_{s,n+1} (x^{(\beta - \varepsilon_s + \varepsilon_{s+1})}) \\
&= q^{-\sum_{i=s+1}^n \beta_i + 1} [\beta_s + 1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) q^{-\beta_s - 1 + \beta_{s+1}} [\beta_{s+1} + 1] (x^{(\beta + \varepsilon_{s+1})}) \\
&\quad - q^{-\beta_s + \beta_{s+1}} [\beta_{s+1} + 1] q^{-\sum_{i=s+1}^n \beta_i - 1} [\beta_s] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta - \varepsilon_s + \varepsilon_{s+1}) [(\beta - \varepsilon_s + \varepsilon_{s+1})_i] \right) (x^{(\beta + \varepsilon_{s+1})}) \\
&= q^{-\sum_{i=s+2}^n \beta_i} [\beta_{s+1} + 1] \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) (x^{(\beta + \varepsilon_{s+1})}) \\
&\quad + q^{-\beta_s + \beta_{s+1}} [\beta_{s+1} + 1] q^{-\sum_{i=s+1}^n \beta_i - 1} [\beta_s] \\
&\quad \cdot \left(\left(\sum_{i=1}^n \theta(\varepsilon_i, \beta) [\beta_i] \right) - \left(\sum_{i=1}^n \theta(\varepsilon_i, \beta - \varepsilon_s + \varepsilon_{s+1}) [(\beta - \varepsilon_s + \varepsilon_{s+1})_i] \right) \right) (x^{(\beta + \varepsilon_{s+1})}) \\
&= e_{s+1,n+1}(x^{(\beta)}) = RHS(x^{(\beta)}).
\end{aligned}$$

Similarly, one can prove (2), by formulas (3.1), (3.9) and (3.10). \square

3.4. We proceed to show Theorem 3.3.

Proof. It suffices to show (1). (2) can be proved in the same manner.

From Proposition 1.4, we conclude that the assertion is true for $1 \leq i, j \leq n$. Therefore, we need only to deal with the case $j = n+1$, namely, $e_{\alpha_{s,n+1}} := e_{s,n+1}$.

For $k = 1$, since T_i are automorphisms of $U_q(sl_{n+1})$, it follows from Proposition 1.4, Lemma 3.1 and formula (3.3) that

$$\begin{aligned}
e_{\alpha_{1,n+1}} &= T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-1} \cdots T_2 T_1(e_n) \\
&= T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-2} \cdots T_2 T_1 T_{n-1}(e_n) \\
&= T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-2} \cdots T_2 T_1([e_{n-1}, e_n]_q) \\
&= [T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-2} \cdots T_2 T_1(e_{n-1}), e_n]_q \\
&= [e_{1,n}, e_n]_q = e_{1,n+1}.
\end{aligned}$$

Assume that the claim holds for j , that is to say,

$$e_{\alpha_{j,n+1}} = T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-1} \cdots T_2 T_1 T_n T_{n-1} \cdots T_{n-j+2}(e_{n-j+1}) := e_{j,n+1}.$$

We will prove it for $j+1$. Using Lemma 3.1 and Lemma 3.4, we obtain

$$\begin{aligned} e_{\alpha_{j+1,n+1}} &= T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-1} \cdots T_2 T_1 T_n T_{n-1} \cdots T_{n-j+2} T_{n-j+1}(e_{n-j}) \\ &= T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-1} \cdots T_2 T_1 T_n T_{n-1} \cdots T_{n-j+2}([e_{n-j+1}, e_{n-j}]_q) \\ &= [T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-1} \cdots T_2 T_1 T_n T_{n-1} \cdots T_{n-j+2}(e_{n-j+1}), \\ &\quad T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-1} \cdots T_2 T_1 T_n T_{n-1} \cdots T_{n-j+2}(e_{n-j})]_q \\ &= [e_{j,n+1}, T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-1} T_{n-2} \cdots T_{n-j-1}(e_{n-j})]_q \\ &= [e_{j,n+1}, T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{n-2} T_{n-3} \cdots T_{n-j-2}(e_{n-j-1})]_q \\ &= \cdots \cdots \\ &= [e_{j,n+1}, T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_{j+1} T_j \cdots T_2 T_1(e_2)]_q \\ &= [e_{j,n+1}, T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_j T_{j-1} \cdots T_1(e_1)]_q \\ &= [e_{j,n+1}, T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_j T_{j-1} \cdots T_2(-f_1 K_1^{-1})]_q \\ &= [e_{j,n+1}, T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_j T_{j-1} \cdots T_2(-f_1) \cdot T_1 T_2 T_1 T_3 T_2 T_1 \cdots T_j T_{j-1} \cdots T_2(K_1^{-1})]_q \\ &= [e_{j,n+1}, -f_j K_j^{-1}]_q = [e_{j,n+1}, T_j(e_j)]_q = e_{j+1,n+1}. \end{aligned}$$

This completes the proof. \square

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